Verifying Probabilistic Programs with Separation Logic Part 3

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Based on joint work with: Alejandro Aguirre, Philipp G. Haselwarter, Kwing Hei-Li, Markus de Medeiros, Puming Liu, Alex Bai, Simon Oddershede Gregersen, and Lars Birkedal

Course page: clutch-project.org/epit2026.html
Rocq code: github.com/logsem/clutch/tree/epit2026

Previous Part

- **Eris**: error credits $↯(\epsilon)$ for "up-to-bad" error bounds reasoning.
- **Eris**$_t$: error-credit induction for almost-sure termination.

This Time

- **Continuous-Eris**: reasoning about exact samplers for continuous distributions using computable reals.
- **Clutch**: program logic for contextual equivalence of higher-order probabilistic programs
- Both use an idea called local presampling tapes that is a "hybrid" of sampling and distribution semantics.

A word on pseudorandomness

"Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin."

— John von Neumann (1951)

A word on floating-point

"Numerical data piles up and numerical programs grow ever more ambitious and complicated while their users become, on average, far less knowledgeable about numerical error-analysis, though no less clever than their predecessors about subjects they care to learn. Consequently numerical anomalies go mostly unobserved or, if observed, routinely misdiagnosed. Fortunately most of them don't matter. Most computations don't matter."

— W. Kahan, *How Futile are Mindless Assessments of Roundoff in Floating-Point Computation?*

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Differential privacy: Given some private data D, compute $f(D)$ and output $f(D) + Y$, where $Y \sim \mathrm{Lap}(1/\epsilon)$. Output is ϵ-differentially private.

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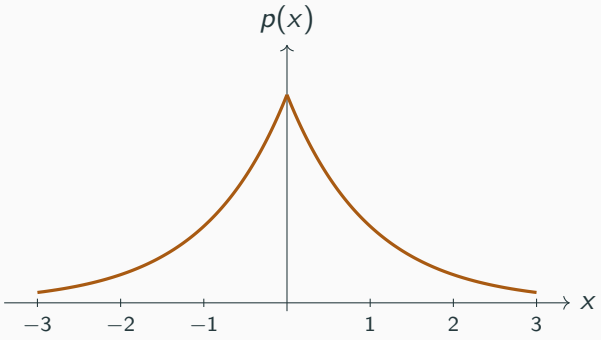

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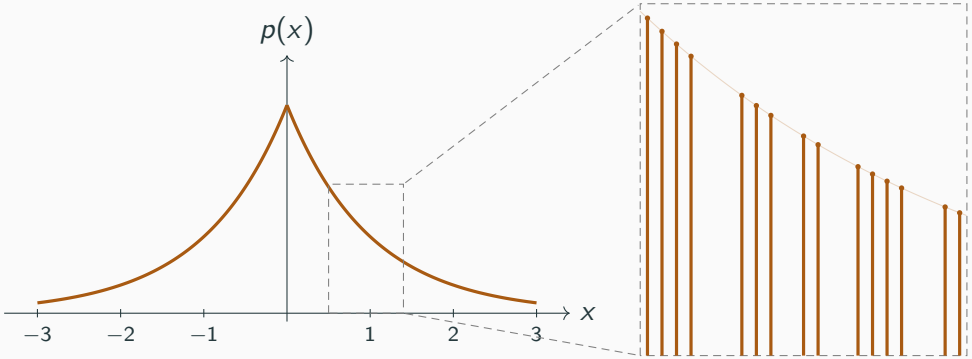

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With standard floating-point Laplace samplers, gaps in the support of $f(D) + Y$ depend on $f(D)$: neighboring inputs $f(D), f(D')$ produce different supports in the output distribution.

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By inspecting the **low-order bits** of a single released value, an attacker recovers $f(D)$ exactly with high probability.

$\Rightarrow$ The textbook floating-point implementation provides no differential privacy.

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Forsake continuous distributions:
use discrete Gaussians, Laplacians, etc.

No roundoff.

Ironically: privacy math sometimes becomes much harder. (Fourier analysis needed for standard results.)

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(One Form of) Computable Reals

Idea: represent a real number r by a function $s_r : \mathbb{Z} \to \mathbb{Z}$ such that, on input p,

$$|s_r(p) - r \cdot 2^{-p}| \leq 1 \quad \text{for every } p \in \mathbb{Z}.$$

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Example. For $r = 4.12345031\ldots$:

p	$r \cdot 2^{-p}$	$s_r(p)$
0	$4.12345\ldots$	4
-4	$65.9752\ldots$	66
-8	$1055.6032\ldots$	1056
-16	$270234.4197\ldots$	270234

Operations on Computable Reals

Arithmetic on reals = higher-order programs that consume and produce approximation functions. Addition:

$$\begin{aligned} \mathit{add}\ r_1\ r_2\ p \triangleq\ & \mathsf{let}\ z = r_1\ (p{-}2) + r_2\ (p{-}2)\ \mathsf{in} \\ & (z \operatorname{div} 4) + (z \bmod 4) \operatorname{div} 2. \end{aligned}$$

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Comparison: query at p, refine if too close to call:

$$\begin{aligned} \mathit{cmp}\ r_1\ r_2\ p \triangleq\ & \mathsf{let}\ n_1 = r_1\ p\ \mathsf{in}\ \mathsf{let}\ n_2 = r_2\ p\ \mathsf{in} \\ & \mathsf{if}\ n_1 + 2 < n_2\ \mathsf{then}\ -1\ \mathsf{else} \\ & \mathsf{if}\ n_2 + 2 < n_1\ \mathsf{then}\quad 1\ \mathsf{else} \\ & \mathit{cmp}\ r_1\ r_2\ (p{-}1). \end{aligned}$$

(Diverges when $r_1 = r_2$.)

Sampling a uniform $r \in [0, 1]$

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$\mathit{GetBits}\ \ell\ N\ A \triangleq$

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$ForceNext\ \ell \triangleq$ match ! ℓ with

  Some$(b, \ell') \Rightarrow (b, \ell')$

  | None $\Rightarrow$ let $b =$ rand 1 in

  let $\ell' =$ ref None in

  $\ell \leftarrow$ Some(b, ℓ');

  (b, ℓ')

  end

Exact Samplers for Other Continuous Distributions

Von Neumann (1952) – exponential distribution (Same 3 page paper where he describes rejection sampling based on densities)

Karney (2016) – Gaussian distribution

Correctness?

How can one prove the correctness of these samplers?

Not so bad in a "nice" lazy probabilistic language (like LazyPPL)?

But what about in an implementation in ML-like language, need:

- Mutable state
- Higher-order functions
- General recursion
- Probabilistic choice

Can we use Eris?

Recall: Marionneau et al. showed that to prove f samples from a distribution discrete D, it suffices to prove in Eris that:

$$\frac{\mathbb{E}_{x\sim D}[\epsilon(x)] = \epsilon_0}{\{\lightning(\epsilon_0)\}\ f\ ()\ \{x.\ \lightning(\epsilon(x))\}}$$

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Idea: generalize this to the continuous case:

$$\frac{\int p(x) \cdot \epsilon(x) dx = \epsilon_0}{\{\lightning(\epsilon_0)\}\ f\ ()\ \{v.\ \exists x.\ \mathsf{IsReal(v,\ x)} * \lightning(\epsilon(x))\}}$$

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then

$$\Pr[\mathit{checker}\ f\ B\ C \neq 1] \;\leq\; \int_{-\infty}^{B/2^C} p(x)\, dx$$

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$$\Pr[\mathit{checker}\ f\ B\ C \neq 1] \ \leq \ \int_{-\infty}^{B/2^C} p(x)\, dx$$

With almost-sure termination, equality holds. Dyadic rationals are dense, so this characterizes the CDF.

Challenge

If we try to apply this to our uniform sampler:

$$U\ \_ \triangleq (\mathsf{ref}\ \mathsf{None})$$

we would have to prove:

$$\frac{\int_0^1 1 \cdot \epsilon(x)dx = \epsilon_0}{\{\lightning(\epsilon_0)\}\ U\ ()\ \{v.\, \exists x.\, \mathsf{IsReal(v,\ x)} * \lightning(\epsilon(x))\}}$$

But this code hasn't even done anything randomized!

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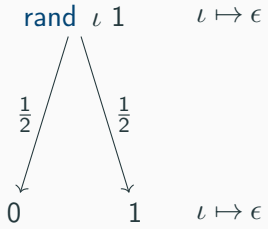
Presampling tapes

$\mathsf{rand}\ \iota\ 1 \qquad \iota \mapsto \epsilon$

Presampling tapes



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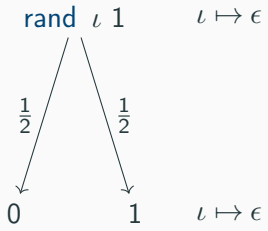

rand ι 1 $\quad \iota \mapsto$ | b | b_1 | b_2 | $\cdots$ |

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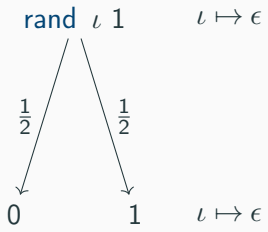

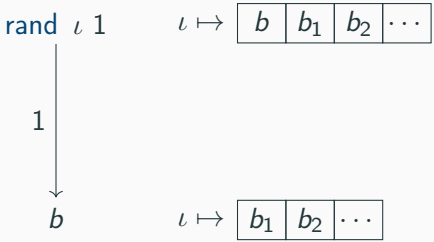

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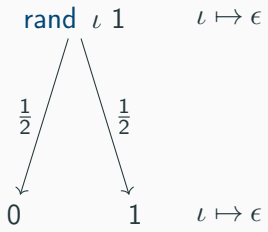

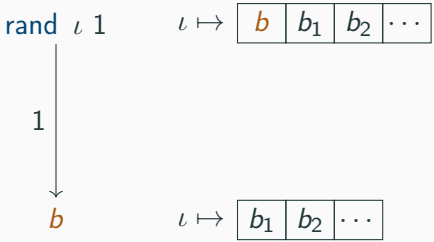

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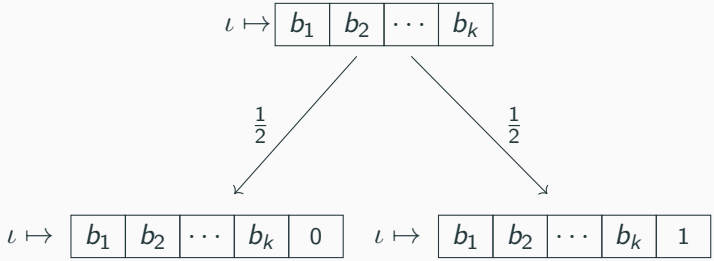

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Introduce a separation logic resource

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$$\frac{\{\exists n.\ \alpha \hookrightarrow (N, \vec{n} \cdot n) * \lightning(\epsilon(n)) * P\}\ e\ \{Q\}}{\{\alpha \hookrightarrow (N, \vec{n}) * \lightning(\mathbb{E}_{\mathsf{Unif}(N)}[\epsilon]) * P\}\ e\ \{Q\}}$$

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It turns reasoning about probabilistic choice into reasoning about state.

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Now to reason about U we can try presample all the samples it needs when it is allocated.

But to stay in the discrete operational world tapes are finite lists of samples. (Why?)

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Problem: uniform sampler U could access an unbounded number of samples.

From Finite to Infinite: Internalize Partiality

Key idea: we work in the **partial** variant of Eris.

We can internalize this partiality using an assertion $\mathsf{stepsLeft}(k)$ saying "in the execution we're reasoning about, the program will run for at most k more steps":

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Each program step decrements stepsLeft.

Proving the Spec for U

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3. By Riemann integrability, $\exists k_2 > 0$ such that the integral is within ϵ' of any Riemann sum using partitions of width $< 1/2^{k_2}$.
4. Set $N = \max(k_1, k_2)$ and pre-sample N bits, using credit averaging so that when we draw $b_1, \ldots, b_N$, we end up with

   $$\lightning\left(b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \cdots + b_N \cdot 2^{-N}\right)$$

   The expected value sum is a Riemann sum that is within ϵ' of ϵ_0.

From U to Beyond

Once you have exact uniform samples and verify the computable real operations, can verify other samplers "just" by doing some integration.

Examples in paper:

- Beta(2,1)
- Exponential
- Gaussian

Clutch: Relational Reasoning

Motivating example

$$\text{let } b = \text{flip in}$$

$$\lambda\_.\, b$$

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$$\begin{array}{l}\mathsf{let}\ b = \mathsf{flip}\ \mathsf{in}\\ \lambda\_.\ b\end{array}$$

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While this example seems esoteric, the pattern shows up in many places: cryptographic security, hash functions, lazily-sampled big integers, ...

What does equivalence mean formally?

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To define this, we formalized **well-typed contexts**:

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To define this, we formalized **well-typed contexts**:

1. Define contexts $\mathcal{C}$ – "program with a hole in it"
2. Define filling operation $\mathcal{C}[e]$ – "plug e in hole in $\mathcal{C}$"
3. Extend type system to types on contexts:

$$\mathcal{C} : (\Gamma \vdash \tau) \Rightarrow (\emptyset \vdash \tau')$$

"if e has type $\Gamma \vdash e : \tau$, then $\emptyset \vdash \mathcal{C}[e] : \tau$"

Contextual equivalence and contextual refinement: classical case

Next, define contextual refinement.

Classical, **non-probabilistic case**:

$$\Gamma \vdash e_1 \precsim_{\text{ctx}} e_2 : \tau \quad \triangleq \quad \forall \tau', (\mathcal{C} : (\Gamma \vdash \tau) \Rightarrow (\emptyset \vdash \text{bool})), \sigma, b.$$
$$(\mathcal{C}[e_1], \sigma) \Downarrow b \Rightarrow \mathcal{C}[e_2], \sigma) \Downarrow b$$

Contextual equivalence and contextual refinement: classical case

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or equivalently:

$$\Gamma \vdash e_1 \precsim_{\text{ctx}} e_2 : \tau \quad \triangleq \quad \forall \tau', (\mathcal{C} : (\Gamma \vdash \tau) \Rightarrow (\emptyset \vdash \tau')), \sigma. \\ (\Downarrow \mathcal{C}[e_1], \sigma) \Rightarrow (\Downarrow \mathcal{C}[e_2], \sigma)$$

Contextual equivalence $\Gamma \vdash e_1 \simeq_{\text{ctx}} e_2 : \tau$ defined as refinement in both directions.

Contextual equivalence and contextual refinement: probabilistic case

For probabilistic programs, generalize the second definition to talk about lower bound of termination probability:

$$\Gamma \vdash e_1 \precsim_{\text{ctx}} e_2 : \tau \quad \triangleq \quad \forall \tau', (\mathcal{C} : (\Gamma \vdash \tau) \Rightarrow (\emptyset \vdash \tau')), \sigma. \\ (\Downarrow \mathcal{C}[e_1], \sigma) \leq (\Downarrow \mathcal{C}[e_2], \sigma)$$

$\Gamma \vdash e_1 \simeq_{\text{ctx}} e_2 : \tau$ is again defined as refinement in both directions.

Proving contextual equivalence is hard

Contextual equivalence quantifies over **all** well-typed contexts $\mathcal{C}$:

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- How can we reason about all such $\mathcal{C}$?
- How do we exploit the typing assumption?

Logical Relations

Logical relations are a standard technique for proving contextual equivalences.

Define a **semantic model** of a typing system:

- Unary case: "what does it mean for a program e to have type τ"
- Binary case: "what does it mean for two programs e_1 and e_2 to be related/equivalent at type τ"

How to build a logical relation?

Defining a logical relation "with your bare hands" is not easy for a language with:

- higher-order state
- recursive types
- probabilistic choice

Program Logic Approach to Logical Relations

A general recipe for logical relations for "hard" languages.

For **unary** logical relations:

1. Start with an expressive Hoare logic $\{P\}\ e\ \{Q\}$ for this language.

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3. Lift to interpretation $[\![x_1 : \tau_1, \ldots, x_n : \tau_n \vdash e : \tau]\!]$ in terms of Hoare triple:

$$\forall v_1, \ldots, v_n.\ \{[\![v_1 : \tau_1]\!] * \cdots * [\![v_n : \tau_n]\!]\}\ e[v_1/x_1] \cdots [v_n/x_n]\ \{x.[\![x : \tau]\!]\}$$

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4. Prove fundamental lemma: if $\Gamma \vdash e : \tau$ then $[\![\Gamma \vdash e : \tau]\!]$

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4. Prove fundamental lemma: if $\Gamma \vdash e : \tau$ then $[\![\Gamma \vdash e : \tau]\!]$
5. Deduce property of interest from soundness of Hoare logic.

Relational Hoare Logic

To apply this recipe for binary logical relations, we need a **relational program logic**.

For non-probabilistic programs, Benton proposed relational hoare logic (RHL):

$$\{P\}\ e_1 \sim e_2\ \{x_1, x_2.Q\}$$

Now P, Q are **relations** on program states.

Read: if we execute e_1 and e_2 in states related by P, and they terminate with values v_1, v_2, then their terminal states will be related by $Q[v_1/x_1][v_2/x_2]$

Probabilistic Relational Hoare Logic

Barthe et al. introduced probabilistic Relational Hoare Logic (pRHL)

Extends RHL with so-called coupling rule:

$$\frac{f : \mathsf{Bool} \to \mathsf{Bool} \text{ is a bijection}}{\{\mathsf{True}\}\ \mathsf{flip} \sim \mathsf{flip}\ \{v_1, v_2.\ v_1 = f(v_2)\}}\ \textsc{pRHL-couple}$$

Why is it called the coupling rule?

A coupling of distributions $\mu_1 \in \mathcal{D}(A)$ and $\mu_2 \in \mathcal{D}(B)$ is a distribution $\mu \in \mathcal{D}(A \times B)$ such that:

- $\sum_{b \in B} \mu(a, b) = \mu_1(a)$ for all $a \in A$
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i.e. the marginals of μ are μ_1 and μ_2.

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The pRHL coupling rule is constructing a coupling between the two flip.

Soundness theorem of pRHL

Recall from Part 1 the evaluation distribution: $(e, \sigma) \Downarrow \mu$ with $\mu \in \mathcal{D}(\mathit{Cfg})$.

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Soundness: if $\{P\}\ e_1 \sim e_2\ \{v_1, v_2.\ Q\}$ holds, then for all $(\sigma_1, \sigma_2) \in P$, whenever $(e_1, \sigma_1) \Downarrow \mu_1$ and $(e_2, \sigma_2) \Downarrow \mu_2$:

$$\mu_1 \sim \mu_2 : R_Q$$

where $R_Q = \{((v_1, \sigma_1'), (v_2, \sigma_2')) \mid (\sigma_1', \sigma_2') \in Q[v_1/x_1][v_2/x_2]\}$.

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i.e. a derivable pRHL quadruple yields an R_Q-coupling of the two evaluation distributions.

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An R-coupling lets us transfer a relational property from paired samples to the full distributions.

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Most importantly, take R to be equality:

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So if pRHL gives us $\{P\}\ e_1 \sim e_2\ \{v_1, v_2.\ v_1 = v_2\}$, we can conclude that e_1 and e_2 produce the same output distribution.

Limitation of the Coupling Rule

The pRHL coupling rule requires us to "synchronize" the probabilistic choices:

$$\frac{f : \mathsf{Bool} \to \mathsf{Bool} \text{ is a bijection}}{\{\mathsf{True}\}\ \mathsf{flip} \sim \mathsf{flip}\ \{v_1, v_2.\ v_1 = f(v_2)\}}\ \textsc{pRHL-couple}$$

Motivating example – Synchronous Couplings Insufficient

$$\texttt{let } b = \texttt{flip in}\quad \lambda\_.\, b \qquad \simeq$$

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```
let r = ref(None) in
λ_. match !r with
      Some(b) ⇒ b
    | None    ⇒ let b = flip in
                r ← Some(b);
                b
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```

Clutch – Relational Reasoning with Asynchronous Couplings

Clutch is a logic that allows proving contextual equivalence of

... probabilistic programs written in an expressive programming language

... using higher-order separation logic,

... and asynchronous probabilistic couplings

More specifically:

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More specifically:

1. A probabilistic relational separation logic on top of Iris
2. A logical refinement judgment (a "logical" logical relation)

$$\Gamma \vDash e_1 \precsim e_2 : \tau$$

that implies contextual refinement.

Refinement judgment

The judgment

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should be read as "in env. Γ, expression e_1 logically refines expression e_2 at type τ".

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Theorem (Fundamental)
If $\Gamma \vdash e : \tau$ *then* $\Gamma \vDash e \precsim e : \tau$.

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Theorem (Fundamental)
If $\Gamma \vdash e : \tau$ *then* $\Gamma \vDash e \precsim e : \tau$.

Theorem (Soundness)
If $\Gamma \vDash e_1 \precsim e_2 : \tau$ *then* $\Gamma \vdash e_1 \precsim_{ctx} e_2 : \tau$.

Symbolic execution rules

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$$\frac{e_1 \overset{\text{pure}}{\rightsquigarrow} e_1' \qquad \Gamma \vDash K[e_1'] \precsim e_2 : \tau}{\Gamma \vDash K[e_1] \precsim e_2 : \tau}$$

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Asynchronous couplings

With presampling tapes, we can synchronously couple tape samplings with program samplings

$$\frac{\iota \hookrightarrow \vec{b} \qquad \forall b.\, \iota \hookrightarrow \vec{b} \cdot b \twoheadrightarrow \Gamma \vDash e \precsim K[b] : \tau}{\Gamma \vDash e \precsim K[\,\mathsf{flip}\,] : \tau}$$

to couple program samplings asynchronously!

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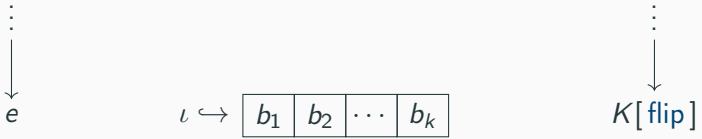
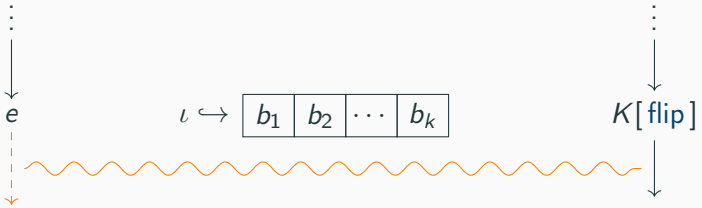

e
ι ↪
b1
b2
· · ·
bk
K [flip]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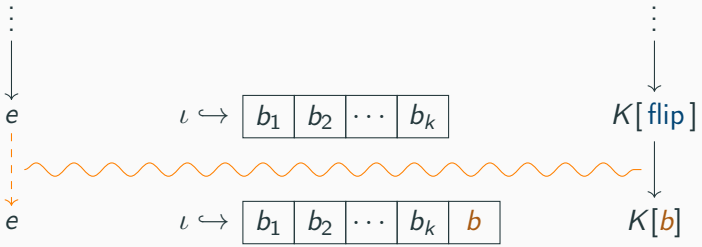
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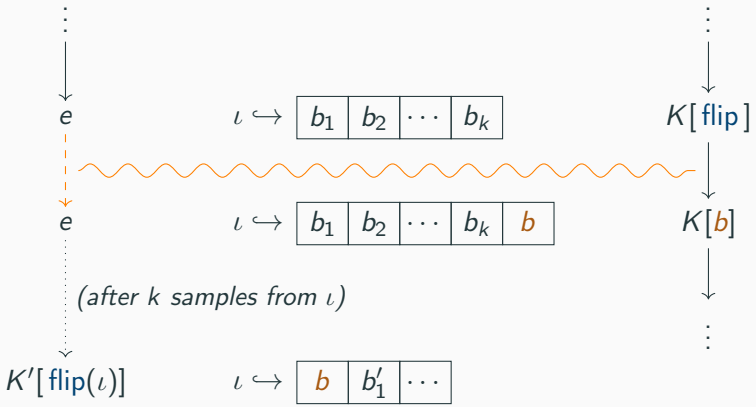
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(after k samples from ι)
K′[flip(ι)]
ι ↪
b1 b2 · · · bk
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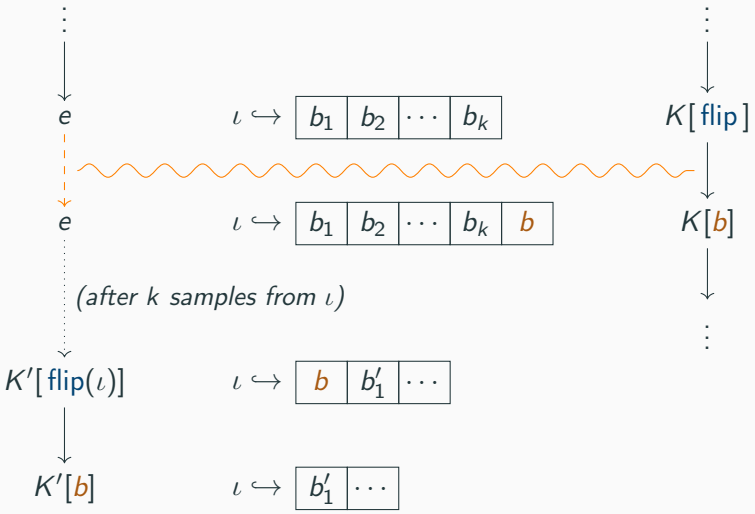
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Motivating example cont'd

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\quad \mathsf{end}
\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
\mathsf{let}\ b = \mathsf{flip}\ \mathsf{in} \\
\lambda\_.\ b
\end{array}
$$

Motivating example cont'd

$$
\begin{array}{l}
\mathsf{let}\ r = \mathsf{ref}(\mathsf{None})\ \mathsf{in} \\
\lambda\_.\ \mathsf{match}\ !\, r\ \mathsf{with} \\
\quad \mathsf{Some}(b) \Rightarrow b \\
\quad \mid \mathsf{None} \Rightarrow \mathsf{let}\ b = \mathsf{flip}\ \mathsf{in} \\
\qquad\qquad\quad r \leftarrow \mathsf{Some}(b); \\
\qquad\qquad\quad b \\
\quad \mathsf{end}
\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
\mathsf{let}\ \iota = \mathsf{tape}\ \mathsf{in} \\
\mathsf{let}\ r = \mathsf{ref}(\mathsf{None})\ \mathsf{in} \\
\lambda\_.\ \mathsf{match}\ !\, r\ \mathsf{with} \\
\quad \mathsf{Some}(b) \Rightarrow b \\
\quad \mid \mathsf{None} \Rightarrow \mathsf{let}\ b = \mathsf{flip}(\iota)\ \mathsf{in} \\
\qquad\qquad\quad r \leftarrow \mathsf{Some}(b); \\
\qquad\qquad\quad b \\
\quad \mathsf{end}
\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
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$$

Motivating example cont'd

$$
\begin{array}{l}
\mathsf{let}\ r = \mathsf{ref}(\mathsf{None})\ \mathsf{in} \\
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\quad \mathsf{Some}(b) \Rightarrow b \\
\quad \mid \mathsf{None} \Rightarrow \mathsf{let}\ b = \mathsf{flip}\ \mathsf{in} \\
\qquad\qquad\quad r \leftarrow \mathsf{Some}(b); \\
\qquad\qquad\quad b \\
\quad \mathsf{end}
\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
\mathsf{let}\ \iota = \mathsf{tape}\ \mathsf{in} \\
\mathsf{let}\ r = \mathsf{ref}(\mathsf{None})\ \mathsf{in} \\
\lambda\_.\ \mathsf{match}\ !\, r\ \mathsf{with} \\
\quad \mathsf{Some}(b) \Rightarrow b \\
\quad \mid \mathsf{None} \Rightarrow \mathsf{let}\ b = \mathsf{flip}(\iota)\ \mathsf{in} \\
\qquad\qquad\quad r \leftarrow \mathsf{Some}(b); \\
\qquad\qquad\quad b \\
\quad \mathsf{end}
\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
\mathsf{let}\ b = \mathsf{flip}\ \mathsf{in} \\
\lambda\_.\ b
\end{array}
$$

Further examples:

- Cryptographic security (ElGamal)
- Lazy vs. Eager Hashing
- Lazily Sampled Big Integers
- Example from Sangiorgi and Vignudelli
- ...

Approximate Equivalences

By extending Clutch with error credits, we can prove approximate equivalences:

- Approxis: error credits to prove total variation bounds
- Clutch-DP: two forms of error credits to prove (ϵ, δ) differential privacy